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## Non-local properties of multi-particle density matrices

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## Abstract

As far as entanglement is concerned, two density matrices of  $n$  particles are equivalent if they are on the same orbit of the group of local unitary transformations,  $U(d_1) \times \cdots \times U(d_n)$  (where the Hilbert space of particle  $r$  has dimension  $d_r$ ). We show that for  $n$  greater than or equal to two, the number of independent parameters needed to specify an  $n$ -particle density matrix up to equivalence is  $\prod_r d_r^2 - \sum_r d_r^2 + n - 1$ . For  $n$  spin- $\frac{1}{2}$  particles we also show how to characterise generic orbits, both by giving an explicit parametrisation of the orbits and by finding a finite set of polynomial invariants which separate the orbits.

In this paper we take some further steps towards understanding multi-particle entanglement by analysing the non-local properties of density matrices of  $n$  particles. This continues the programme, begun in [1], in which we gave a framework for studying the space of pure states of  $n$  spin-1/2 particles. As discussed in [1], the space of pure states of  $n$  spin-1/2 particles is  $\mathbb{C}^{2^n} = \mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2$ ; however not all the  $2^n$  complex parameters have non-local significance: the group of local transformations,  $U(2)^n$  acts on the space of states and two states which may be reached from each other by local actions are equivalent as far as their non-local properties are concerned. Each equivalence class of locally equivalent density matrices is an *orbit* of this group. For many purposes, only parameters describing non-local properties are significant; an example is that any good measure of entanglement must be invariant under local transformations, and thus it should be a function of non-local parameters only (here and henceforth we will refer to parameters which are invariant under local transformations as invariants). A key question is to identify the invariants.

In this paper we will focus on density matrices and show that for  $n \geq 2$ , of the  $2^{2n} - 1$  real parameters describing density matrices of  $n$  spin-1/2 particles  $2^{2n} - 3n - 1$  are invariant under local transformations,  $U(2)^n$ . This generalises to an arbitrary set of  $n$  particles as  $\prod_r d_r^2 - \sum_r d_r^2 + n - 1$  where  $d_r$  is the dimension of the state space of the  $r$ th particle. For  $n$  spin- $\frac{1}{2}$  particles we also show how to characterise generic orbits, both by giving an explicit parametrisation of the orbits and by finding a finite set of polynomial invariants which separate the orbits. Thus given two density matrices we can compute explicitly whether they are on the same orbit or not. Other authors have also discussed the use of invariants in discussing entanglement [2] [3] and applied invariant theory to quantum codes [4].

In order to calculate the number of functionally independent invariants it will be convenient to find the dimension of the orbit of a generic density matrix under the group of local transformations. The dimension of the orbit is the number of parameters describing the location of a density matrix on the orbit. The total number of parameters ( $2^{2n} - 1$  real parameters) describing the space of density matrices minus the number of parameters describing a generic orbit (the dimension of the orbit) gives the number of parameters

describing the location of the orbit in the space of orbits, i.e. the number of parameters describing the non-local properties of the density matrices.

To fix notation, it will be convenient to consider the case of a one-particle density matrix first. The space of pure states of a single spin-1/2 particle is  $\mathbb{C}^2$  and thus a density matrix is a  $2 \times 2$  complex matrix which is hermitian, positive and with trace one, and may therefore be described by three real parameters. A particularly convenient representation of such a matrix is

$$\rho = \frac{1}{2} \mathbb{1}_2 + \alpha_i \sigma_i, \quad (1)$$

where  $\alpha_i$ ,  $i = 1, 2, 3$  are real and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \mathbb{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (2)$$

We note that

$$\sigma_i \sigma_j = i \epsilon_{ijk} \sigma_k + \delta_{ij}. \quad (3)$$

Under a local transformation by a unitary matrix  $U$ ,  $\rho$  is transformed as

$$\rho \mapsto U \rho U^\dagger. \quad (4)$$

The group  $U(2)$  is isomorphic to  $U(1) \times SU(2)$  where, physically, the  $U(1)$  is the phase transformation of a state, represented by a unitary matrix  $e^{i\phi} \mathbb{1}_2$ . This element clearly leaves any density matrix invariant under the transformation (4) so that when considering the action (4) we may restrict attention to elements of  $SU(2)$ . In order to find the number of invariants it will be more convenient to find the dimension of a generic orbit under the action of  $SU(2)$ . To do so one may work infinitesimally. Thus, associated to the action of the Lie algebra of the group of local transformations acting on the space of density matrices there is a vector field: if we take an element  $T$  of a basis for the Lie algebra, the action of the group element  $k = \exp i\epsilon T \in K$  on an element  $\rho$  induces an action on functions from  $\rho$  to  $\mathbb{C}$ ; and the vector field,  $X_T$ , associated to the Lie algebra element  $T$  is found by differentiating:

$$X_T f(v) \stackrel{\text{def}}{=} \frac{\partial}{\partial \epsilon} f(e^{i\epsilon T} \rho) |_{\epsilon=0} = \frac{\partial}{\partial \epsilon} f(\rho + \delta \rho) |_{\epsilon=0}. \quad (5)$$

The linear span of vector fields at the point  $\rho$  associated with the whole Lie algebra forms the tangent space to the orbit at the point  $\rho$  and so the number of linearly independent vector fields at this point gives the dimension of the orbit.

A general element of the Lie algebra in the spin-1/2 representation is given by

$$T = \eta_i \sigma_i \quad (6)$$

and its action on the density matrix is to give an infinitesimal transformation

$$\delta \rho = i[T, \rho] \quad (7)$$

where  $[,]$  is the matrix commutator.

We may therefore calculate the three vector fields  $X_1, X_2$  and  $X_3$  associated to the Lie algebra elements  $\sigma_1, \sigma_2$  and  $\sigma_3$  as

$$X_1 = \alpha_2 \frac{\partial}{\partial \alpha_3} - \alpha_3 \frac{\partial}{\partial \alpha_2}, \quad X_2 = \alpha_3 \frac{\partial}{\partial \alpha_1} - \alpha_1 \frac{\partial}{\partial \alpha_3}, \quad X_3 = \alpha_1 \frac{\partial}{\partial \alpha_2} - \alpha_2 \frac{\partial}{\partial \alpha_1}. \quad (8)$$

We note that at generic values of  $\alpha_1, \alpha_2, \alpha_3$  only two of these vector fields are linearly independent since

$$\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 = 0. \quad (9)$$

Thus the dimension of the generic orbit is two and therefore of the three parameters describing a generic density matrix, two are non-invariant leaving only one invariant parameter, as one expects since only the single independent eigenvalue of  $\rho$  is invariant under local transformations.

We note that the effect of the transformations (4) is to act on the vector  $\alpha$  by rotation by an orthogonal matrix, i.e. an element of  $SO(3)$  - this follows from the fact that  $\alpha_i \sigma_i$  is the representative of a Lie algebra element and the conjugation action (4) is the adjoint action of the group on its Lie algebra. We may thus find a way of exhibiting the invariant under local transformations:

$$I = \alpha_i \alpha_j \delta_{ij} = |\alpha|^2 \quad (10)$$

where we have used the fact that  $SO(3)$  has an invariant tensor  $\delta^{ij}$ . We note that this invariant may also be expressed as

$$I = \text{Tr}(\rho^2) - \frac{1}{2}. \quad (11)$$

We now turn to the case of two-particle density matrices. Such a density matrix has 15 real parameters, and the maximum dimension that a generic orbit could have is 6 (corresponding to two copies of  $SU(2)$ ) if all the vector fields corresponding to a basis of the Lie algebra were independent. We will show that the vector fields do indeed span 6 dimensions, and thus that there are 9 non-local parameters.

We may write a density matrix as

$$\rho = \frac{1}{4} \mathbf{1}_2 \otimes \mathbf{1}_2 + \alpha_i \sigma_i \otimes \mathbf{1}_2 + \beta_i \mathbf{1}_2 \otimes \sigma_i + R_{ij} \sigma_i \otimes \sigma_j. \quad (12)$$

The action of a Lie algebra element of the subgroup  $SU(2)$  acting on the first component of the tensor product is

$$\begin{aligned} \delta^{(1)} \rho &= i[\eta_k \sigma_k \otimes \mathbf{1}_2, \rho] \\ &= \alpha_k \eta_m \epsilon_{mki} \sigma_i \otimes \mathbf{1}_2 + R_{kj} \eta_m \epsilon_{mki} \sigma_i \otimes \sigma_j, \end{aligned} \quad (13)$$

and that corresponding to a Lie algebra element of the subgroup  $SU(2)$  acting on the second component of the tensor product,

$$\begin{aligned} \delta^{(2)} \rho &= i[\eta_k \mathbf{1}_2 \otimes \sigma_k, \rho] \\ &= \beta_k \eta_m \epsilon_{mki} \mathbf{1}_2 \otimes \sigma_i + R_{ik} \eta_m \epsilon_{mkj} \sigma_i \otimes \sigma_j. \end{aligned} \quad (14)$$

The vector fields corresponding to the six basis elements  $\sigma_i \otimes \mathbf{1}_2$ ,  $\mathbf{1}_2 \otimes \sigma_i$  are

$$\begin{aligned} X_k &= \epsilon_{kim} \left( \alpha_i \frac{\partial}{\partial \alpha_m} + R_{ij} \frac{\partial}{\partial R_{mj}} \right), \\ Y_k &= \epsilon_{kim} \left( \beta_i \frac{\partial}{\partial \beta_m} + R_{ji} \frac{\partial}{\partial R_{jm}} \right) \end{aligned} \quad (15)$$

Consider the set  $X_i$  first: one can see that these three are linearly independent at generic points by considering the coefficients of  $\partial/\partial\alpha_i$ , since a linear relation would have to be of the form  $\alpha_k X_k = 0$ , but one can see that this relation will not hold for non-zero  $\alpha$ 's by looking at the coefficients of the partial derivatives with respect to  $R_{ij}$ . Similarly by considering the coefficients of the partial derivatives with respect to  $\beta_1, \beta_2, \beta_3$ , one sees that  $Y_1, Y_2, Y_3$  are linearly independent. Finally, we note that the coefficients of the partial derivatives with respect to  $\beta_1, \beta_2, \beta_3$  are zero for  $X_1, X_2, X_3$  and the coefficients of the partial derivatives with respect to  $\alpha_1, \alpha_2, \alpha_3$  are zero for  $Y_1, Y_2, Y_3$  so that there can be no linear relation at all between the six vector fields  $X_1, X_2, X_3, Y_1, Y_2, Y_3$ . Thus the dimension of the orbit of a generic density matrix is 6 and thus the number of non-local parameters,  $15 - 6 = 9$ .

In general, we can consider a system of  $n$  particles with individual state spaces of dimensions  $d_1, \dots, d_n$ . The density matrix is a hermitian  $D \times D$  matrix with trace 1, where  $D = d_1 d_2 \dots d_n$ , and therefore requires  $D^2 - 1$  real parameters which can be taken to be the coefficients  $\alpha^{(1)}, \dots, \alpha^{(n)}, \dots, R$  in an expansion

$$\rho = \frac{1}{D} 1_{d_1} \otimes \dots \otimes 1_{d_n} + \sum_{r=1}^n \alpha_{i_r}^{(r)} 1 \otimes \dots \otimes T_{i_r}^{(r)} \otimes \dots \otimes 1 + \dots + R_{i_1 \dots i_n} T_{i_1}^{(1)} \otimes \dots \otimes T_{i_n}^{(n)} \quad (16)$$

where  $T_{i_r}^{(r)}$  ( $i_r = 1, \dots, d_r^2 - 1$ ) are a basis set of traceless hermitian  $d_r \times d_r$  matrices (generators of  $SU(d_r)$ ). The action of an infinitesimal generator of  $SU(d_r)$  acting on the  $r$ th factor of the tensor product is

$$\delta^{(r)} \rho = c_{ijk}^{(r)} \eta_i (\alpha_j^{(r)} 1 \otimes \dots \otimes T_k^{(r)} \otimes \dots \otimes 1 + \dots) \quad (i, j, k = 1, \dots, d_r^2 - 1) \quad (17)$$

where  $c_{ijk}^{(r)}$  are the structure constants of  $SU(d_r)$ . Thus the infinitesimal action of local transformations is given by a set of vector fields

$$X_i^{(r)} = c_{ijk}^{(r)} \left( \alpha_j^{(r)} \frac{\partial}{\partial \alpha_k^{(r)}} + \dots + R_{i_1 \dots j \dots i_n} \frac{\partial}{\partial R_{i_1 \dots k \dots i_n}} \right). \quad (18)$$

Similar considerations to those used above for the case of two spin- $\frac{1}{2}$  particles show that these vector fields are generically all independent. Thus the generic orbit has dimension  $d_1^2 + \dots + d_n^2 - n$ . Since the space of density matrices has dimension  $d_1^2 \dots d_n^2$ , there are a total of

$$\prod_r d_r^2 - \sum_r d_r^2 + n - 1 \quad (19)$$

non-local invariants.

Let us now return to the case of  $n \geq 2$  spin-1/2 particles and explicitly identify a set of invariant parameters which characterise generic orbits. To be explicit, consider the case of three spin-1/2 particles with density matrix which may be written as

$$\begin{aligned} \rho = & \frac{1}{8} \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \mathbf{1}_2 + \alpha_i \sigma_i \otimes \mathbf{1}_2 \otimes \mathbf{1}_2 + \beta_i \mathbf{1}_2 \otimes \sigma_i \otimes \mathbf{1}_2 + \gamma_i \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \sigma_i \\ & + R_{ij} \sigma_i \otimes \sigma_j \otimes \mathbf{1}_2 + S_{ij} \sigma_i \otimes \mathbf{1}_2 \otimes \sigma_j + T_{ij} \mathbf{1}_2 \otimes \sigma_i \otimes \sigma_j \\ & + Q_{ijk} \sigma_i \otimes \sigma_j \otimes \sigma_k. \end{aligned} \quad (20)$$

The action by a local unitary transformation on the first component in the tensor product induces the following transformations on the components of  $\rho$

$$\alpha_i \mapsto L_{ij} \alpha_j; \quad R_{ij} \mapsto L_{ik} R_{kj}; \quad S_{ij} \mapsto L_{ik} S_{kj}; \quad Q_{ijk} \mapsto L_{im} Q_{mjk} \quad (21)$$

where  $L_{ij}$  is an orthogonal matrix, and the other components of  $\rho$  do not change. Similarly actions by a local transformations on the second and third components of the tensor product induce

$$\beta_i \mapsto M_{ij} \beta_j; \quad R_{ij} \mapsto M_{jk} R_{ik}; \quad T_{ij} \mapsto M_{ik} T_{kj}; \quad Q_{ijk} \mapsto M_{jm} Q_{imk} \quad (22)$$

and

$$\gamma_i \mapsto N_{ij} \gamma_j; \quad S_{ij} \mapsto N_{jk} S_{ik}; \quad T_{ij} \mapsto N_{jk} T_{ik}; \quad Q_{ijk} \mapsto N_{km} Q_{ijm} \quad (23)$$

respectively, where  $M_{ij}$  and  $N_{ij}$  are orthogonal matrices independent of  $L$ .

We may fix a canonical point on a generic orbit as follows: firstly let us define

$$X_{ii'} = Q_{ijk} Q_{i'jk}; \quad Y_{jj'} = Q_{ijk} Q_{ij'k}; \quad Z_{kk'} = Q_{ijk} Q_{ijk'}, \quad (24)$$

and perform unitary transformations on particles 1, 2 and 3 so as to move to a point on the orbit in which  $X$ ,  $Y$  and  $Z$  are diagonal; generically the diagonal entries are distinct and we

can arrange them in decreasing order ( $X$ ,  $Y$  and  $Z$  are hermitian, positive matrices). The only remaining transformations which leave  $X$ ,  $Y$  and  $Z$  in these forms are local unitary transformations which induce orthogonal transformations in which  $L_{ij}$ ,  $M_{ij}$  and  $N_{ij}$  are one of the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (25)$$

We may specify a canonical point on the generic orbit uniquely by specifying that all the components of  $\alpha$  have the same sign, and similarly for  $\beta$  and  $\gamma$ . This method works as long as  $X$ ,  $Y$  and  $Z$  have distinct eigenvalues and the components of  $\alpha$ ,  $\beta$  and  $\gamma$  are not zero at the canonical point on the orbit. The parameters which describe the generic orbits are the components of  $\alpha, \beta, \gamma, R, S, T$  and  $Q$  at the canonical point on the orbit. We note that the number of parameters describing the canonical point are the  $2^6 - 1 = 63$  components of  $\alpha, \beta, \gamma, R, S, T$  and  $Q$  minus the  $3 \times 3 = 9$  constraints that the non-diagonal elements of  $X$ ,  $Y$  and  $Z$  are zero; thus the number of non-local parameters is 54 as given by the general formula.

We note that the fact that the canonical point, as constructed, is unique means that all points on the same orbit will have the same canonical representative: conversely, if two density matrices  $\rho_1$  and  $\rho_2$  have the same canonical form, then

$$U_1 \rho_1 U_1^\dagger = \rho_{\text{canonical}} = U_2 \rho_2 U_2^\dagger \quad (26)$$

for some  $U_1$  and  $U_2$ , so that

$$\rho_2 = (U_2^\dagger U_1) \rho_1 (U_2^\dagger U_1)^\dagger \quad (27)$$

and thus  $\rho_1$  and  $\rho_2$  are on the same orbit.

We now describe a *finite* set of polynomial invariants which separate generic orbits by finding a set which allows one to calculate the components of  $\alpha, \beta, \gamma, R, S, T$  and  $Q$  at this canonical point. The complete infinite set of polynomial invariants is found by contracting

the indices of  $\alpha, \beta, \gamma, R, S, T$  and  $Q$  with the invariant tensors  $\delta_{ij}$  and  $\epsilon_{ijk}$ . However we may find a finite set of invariants which separates generic orbits. Firstly we note that  $\text{tr}(X)$ ,  $\text{tr}(X^2)$  and  $\text{tr}(X^3)$  determine the diagonal elements  $\lambda_1^2$ ,  $\lambda_2^2$  and  $\lambda_3^2$  of  $X$ , and similarly for  $Y$  and  $Z$ . Now consider the three invariants  $A_{2n} = \alpha^T X^{n-1} \alpha$ ,  $n = 1, 2, 3$ . We may write these three invariants in the following way:

$$\begin{pmatrix} 1 & 1 & 1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1^4 & \lambda_2^4 & \lambda_3^4 \end{pmatrix} \begin{pmatrix} a_1^2 \\ a_2^2 \\ a_3^2 \end{pmatrix} = \begin{pmatrix} A_2 \\ A_4 \\ A_6 \end{pmatrix}, \quad (28)$$

where  $a_1$ ,  $a_2$  and  $a_3$  are the components of  $\alpha$  at the canonical point on the orbit. The Vandermonde matrix

$$\Lambda = \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1^4 & \lambda_2^4 & \lambda_3^4 \end{pmatrix} \quad (29)$$

has determinant  $(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)(\lambda_3^2 - \lambda_1^2)$ , and we may solve for  $a_1^2$ ,  $a_2^2$  and  $a_3^2$  as long as  $\det \Lambda$  is non-zero. Also if the invariant

$$A_9 = \epsilon_{ijk} \alpha_i (X\alpha)_j (X^2\alpha)_k = \alpha \cdot (X\alpha) \wedge (X^2\alpha) = a_1 a_2 a_3 \det \Lambda \quad (30)$$

is non-zero, then we may determine the sign of the components of  $\alpha$ ; recall that, by definition, all the components of  $\alpha$  have the same sign at the canonical point. The analogous expressions  $B_9, C_9$  determine the values of  $\beta$  and  $\gamma$  at the canonical point. The values of the components of  $R$  at the canonical point may be calculated from the following nine invariants:

$$I_{r,s} = (X^{r-1}\alpha)_i (Y^{s-1}\beta)_j R_{ij}. \quad (31)$$

These nine equations may be put together into a matrix form

$$I = ((\Lambda F) \otimes (MG)) R, \quad (32)$$

where  $I$  and  $R$  are column vectors with nine components and the matrices  $\Lambda$ ,  $F$ ,  $M$  and  $G$  are

$$\Lambda = \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1^4 & \lambda_2^4 & \lambda_3^4 \end{pmatrix}; \quad F = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}; \quad M = \begin{pmatrix} 1 & 1 & 1 \\ \mu_1^2 & \mu_2^2 & \mu_3^2 \\ \mu_1^4 & \mu_2^4 & \mu_3^4 \end{pmatrix}; \quad G = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix}, \quad (33)$$

where  $\mu_1^2$ ,  $\mu_2^2$  and  $\mu_3^2$  are the diagonal elements of  $Y$ . We note that  $\det(\Lambda F) = A_9$  and  $\det(MG) = B_9$ , so since we are assuming that these are non-zero we may invert the matrix equation to find the components  $R_{ij}$ . The components of  $S$  and  $T$  may be found in a similar way. Finally we may use the 27 invariants

$$I_{r,s,t} = (X^{r-1}\alpha)_i(Y^{s-1}\beta)_j(Z^{t-1}\gamma)_k Q_{ijk}. \quad (34)$$

to find the components of  $Q$  at the canonical point on the orbit in terms of the  $I_{r,s,t}$  (there will, of course, be some relations between these components due the constraints that  $X$ ,  $Y$  and  $Z$  are diagonal).

Thus, by showing that the following set of polynomial invariants is sufficient to calculate the components of a generic density matrix at the canonical point we have demonstrated that they characterise generic orbits:

$$\begin{aligned} & \text{tr}X^r, \quad \text{tr}Y^r, \quad \text{tr}Z^r \\ & \alpha^T X^{r-1}\alpha, \quad \beta^T Y^{r-1}\beta, \quad \gamma^T Z^{r-1}\gamma; \\ & \alpha.(X\alpha) \wedge (X^2\alpha), \quad \beta.(Y\beta) \wedge (Y^2\beta), \quad \gamma.(Z\gamma) \wedge (Z^2\gamma) \\ & (X^{r-1}\alpha)_i(Y^{s-1}\beta)_j R_{ij}, \quad (Y^{r-1}\beta)_i(Z^{s-1}\gamma)_j T_{ij}, \quad (X^{s-1}\alpha)_i(Z^{r-1}\gamma)_j S_{ij}; \\ & (X^{r-1}\alpha)_i(Y^{s-1}\beta)_j(Z^{t-1}\gamma)_k Q_{ijk}; \end{aligned} \quad (35)$$

the indices  $r, s, t$  range over the values 1, 2, 3.

If two density matrices have different values of any of these invariants they are not on the same orbit; if they have same value of all of these invariants, and if  $A_9$ ,  $B_9$  and  $C_9$  are non-zero, then the density matrices are locally equivalent.

We note that the number of independent components of a generic density matrix at the canonical point is equal to the number of functionally independent parameters calculated

at the beginning of this letter. However, the number of polynomial invariants needed to characterise the generic orbit is greater than this; this is related to the fact that the ring of invariants is non-polynomial, i.e. that the geometry of the space of orbits is non-trivial.

The procedure given above can be used for all  $n \geq 2$ : use the tensors of highest rank and rank one in the expression for  $\rho$  to fix a canonical point on the orbit; the polynomials which separate the generic orbits are the analogues of those used in the case  $n = 3$ .

In the case of  $n = 2$  this method can be used but there is some redundancy in the description we have given: the matrices  $X_{ii'} = R_{ij}R_{i'j}$  and  $Y_{jj'} = R_{ij}R_{ij'}$  (using the notation of (12)) have the same eigenvalues and the matrix  $R_{ij}$  is diagonal at the canonical point. In this case there are nine functionally independent invariants which specify the squares of the non-zero components of  $\alpha$ ,  $\beta$  and  $R$  at the canonical point on a generic orbit:  $\text{tr}X^n$ ,  $\alpha^T X^{m-1}\alpha$  and  $\beta^T Y^{p-1}\beta$ , where  $n, m, p$  take the values 1, 2, 3. Additional invariants are needed to specify the signs of the non-zero components. The five invariants  $\alpha.(X\alpha) \wedge (X^2\alpha)$ ,  $\beta.(Y\beta) \wedge (Y^2\beta)$  and  $\alpha X^{r-1}R\beta$ ,  $r = 1, 2, 3$ , are sufficient to determine these signs for generic orbits and hence separate these orbits. In fact, using slightly different arguments, one can show that, in this case, one can reduce the number of polynomial invariants to ten, namely  $\text{tr}X$ ,  $\text{tr}X^2$ ,  $\det R$ ,  $\alpha^T X^{r-1}\alpha$ ,  $\alpha^T X^{r-1}R\beta$ ,  $r = 1, 2, 3$  and  $A_9$ , which are subject to a single relation expressing  $A_9^2$  as a function of the other invariants.

The general idea of investigating canonical points on orbits in the way we have described is also appropriate for higher spins, but the situation is somewhat more complicated. Consider the example of two particles of spin one in which case the unitary group under which  $\rho$  transforms is  $SU(3)$ .  $\rho$  may be written as

$$\rho = \frac{1}{64}1_8 \otimes 1_8 + \alpha_i T_i \otimes 1_8 + \beta_i 1_8 \otimes T_i + R_{ij} T_i \otimes T_j \quad (36)$$

where  $T_i$ ,  $i = 1 \dots 8$  are representatives of a basis for the Lie algebra of  $SU(3)$  in the adjoint representation and  $1_8$  is the  $8 \times 8$  identity matrix. However, the adjoint representation of  $SU(3)$  is equivalent not to  $SO(8)$  but to an eight dimensional subgroup of it; this means that we cannot transform  $\rho$  so that  $RR^T$  and  $R^T R$  are diagonal so the canonical form is

rather more complicated than in the case of spin-1/2 particles.

In summary we have shown how to calculate the number of functionally independent parameters needed to determine whether or not two density matrices are locally equivalent. We have also shown how to characterise the generic classes of locally equivalent density matrices of  $n$  spin-1/2 particles by two methods: (a) by finding an explicit set of non-polynomial invariants (the components of the density matrices at the canonical points on the orbits) and (b) by finding an explicit finite set of polynomial invariants. These methods work for generic density matrices; in a future publication we intend to give a systematic method for characterising classes of locally equivalent non-generic density matrices. In particular this will give a basis for the ring of invariants. We note that the canonical point on certain types of non-generic orbit has non-trivial stability group; this is a signature that density matrices on this orbit have special types of entanglement [1].

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## REFERENCES

- [1] N. Linden and S. Popescu, [quant-ph/9711017](#) to be published in *Fortschritte der Physik*.
- [2] J. Schlienz and G. Mahler, *Physics Letters A* 39 (1996).
- [3] M. Grassl, M. Rötteler and T. Beth, [quant-ph/9712040](#).
- [4] E. Rains, [quant-ph/9704042](#)